This article was downloaded by: [University of Toronto Libraries]
On: 24 December 2014, At: 07:10
Publisher: Taylor & Francis
Informa Ltd Registered in England and Wales Registered Number: 1072954
Registered office: Mortimer House, 37-41 Mortimer Street, London W1T 3JH, UK

The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science: Series 7
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tphm18

LXXXVI. The metric of space and mass deficiency

A.M. Mosharrafa Ph.D. D.Sc. [a]
[a] Fuad I. University , Cairo
Published online: 15 Jul 2009.

To cite this article: A.M. Mosharrafa Ph.D. D.Sc. (1948) LXXXVI. The metric of space and mass deficiency, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science: Series 7, 39:296, 728-738, DOI: 10.1080/14786444808521790

To link to this article: http://dx.doi.org/10.1080/14786444808521790

PLEASE SCROLL DOWN FOR ARTICLE

Taylor & Francis makes every effort to ensure the accuracy of all the information (the “Content”) contained in the publications on our platform. However, Taylor & Francis, our agents, and our licensors make no representations or warranties whatsoever as to the accuracy, completeness, or suitability for any purpose of the Content. Any opinions and views expressed in this publication are the opinions and views of the authors, and are not the views of or endorsed by Taylor & Francis. The accuracy of the Content should not be relied upon and should be independently verified with primary sources of information. Taylor and Francis shall not be liable for any losses, actions, claims, proceedings, demands, costs, expenses, damages, and other liabilities whatsoever or howsoever caused arising directly or indirectly in connection with, in relation to or arising out of the use of the Content.

This article may be used for research, teaching, and private study purposes. Any substantial or systematic reproduction, redistribution, reselling, loan, sub-licensing, systematic supply, or distribution in any form to anyone is expressly forbidden. Terms & Conditions of access and use can be found at http://www.tandfonline.com/page/terms-and-conditions

REFERENCES

(1) G. Ascoli, Sul comportamento asintotico degli integrali delle equazioni differenziali lineari di 2° ordine, Atti della Reale Accademia Nazionale dei Lincei, Rendiconti, ser. 6, vol. xxii. pp. 234–243 (1935).

(2) A. Kneser, Untersuchungen und asymptotische Darstellung der Integrale gewisser Differentialgleichungen bei grossen reellen Werthen des Arguments, Zweiter Aufsatz, Journal für die reine und angewandte Mathematik, vol. cxvii. pp. 72–103 (1897).

(3) W. F. Osgood, On a theorem of oscillation, Bulletin of the American Mathematical Society, vol. xxv. pp. 216–221 (1919).

(4) O. Perron, Über ein vermeintliches Stabilitätskriterium, Göttinger Nachrichten, pp. 28–29 (1930).

(5) A. Wintner, Asymptotic integration constants, American Journal of Mathematics, vol. lxviii. pp. 553–559 (1946).

LXXXVI. *The Metric of Space and Mass Deficiency.*

By A. M. MOSHARRAFA, Ph.D., D.Sc.,
Professor of Applied Mathematics in the Fuad I. University, Cairo*.

[Received January 8, 1946.]

§ 1. *Introduction and Aim.*

IN the theory of relativity the invariant quadratic differential form

$$g_{\mu\nu}\,dx^\mu\,dx^\nu\ ;\ \mu, \nu=1, 2, 3, 4, \qquad (1)$$

defines the metric of space-time. The physical interval between two infinitesimally close events at a point is thus made to depend on the fundamental tensor $g_{\mu\nu}$ which defines the gravitational field at the point, but is quite independent of the electromagnetic field. The theory leads to the world-line of a gravitational particle as a geodesic, but fails to obtain the world-line of a charged particle as a geodesic. Attempts have been made at a unified field theory, notably by Weyl [(1)], who introduces a " gauge " factor in the metric, and by Kaluza [(2)], on the basis of a five-dimensional geometry. The aim of the present paper is to show that, by using a slightly generalized form of the Riemannian metric, it is possible to reduce the world-line of a charged particle to a geodesic ; and furthermore, to show that this new theory provides a quantitative explanation of the phenomenon of mass deficiency (sometimes described as the " packing " effect) and a basis for measuring nuclear fields.

§ 2. *Introduction of a New Metric and Equations of a Geodesic.*

If we write the equation of the Riemannian metric, defining the element of length ds, in the form

$$g_{\mu\nu}\,dx^\mu\,dx^\nu - ds^2 = 0,$$

---

* Communicated by the Author.

Downloaded by [University of Toronto Libraries] at 07:10 24 December 2014

we observe that the left-hand side is of a general homogeneous form in the four differentials dx', dx^2, dx^3 and dx^4, but not in the five differentials dx', dx^2, dx^3, dx^4 and ds taken together. This suggests a generalization by the addition of terms in $dx^\mu ds$. We accordingly adopt a metric given by

$$g_{\mu\nu}\,dx^\mu\,dx^\nu + 2f_\mu\,dx^\mu\,ds - ds^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

There is no loss of generality in leaving the coefficient of ds^2 equal to -1, since this coefficient must be an invariant, and hence we can divide by it and include it in the coefficients of the other terms. The metric now contains the tensor $g_{\mu\nu}$ and the covariant vector f_μ. We proceed to obtain the equations of a geodesic. We have from (2), operating with δ by the usual method of variation,

$$dx^\mu\,dx^\nu\,\delta g_{\mu\nu} + g_{\mu\nu}\,dx^\mu\,\delta(dx^\nu) + g_{\mu\nu}\,dx^\nu\,\delta(dx^\mu) + 2dx^\mu\,ds\,\delta f_\mu$$
$$+ 2f_\mu\,ds\,\delta(dx^\mu) + 2f_\mu\,dx^\mu\,\delta(ds) - 2ds\,\delta(ds) = 0.$$

$$\therefore \quad 2\delta(ds)\left[1 - f_\mu \frac{dx^\mu}{ds}\right] = \left\{\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\,\delta x^\sigma + g_{\mu\nu}\frac{dx^\mu}{ds}\frac{d}{ds}(\delta x^\nu) + g_{\mu\nu}\frac{dx^\nu}{ds}\frac{d}{ds}(\delta x^\mu)\right.$$
$$\left. + 2\frac{dx^\mu}{ds}\frac{\partial f_\mu}{\partial x^\sigma}\,\delta x^\sigma + 2f_\mu \frac{d}{ds}(\delta x^\mu)\right\} ds.$$

The condition for a geodesic,

$$\int \delta(ds) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

now gives

$$\frac{1}{2}\int \left\{ \frac{\left[\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{\partial g_{\mu\nu}}{\partial x^\sigma} + 2\frac{ds}{dx^\mu}\frac{\partial f_\mu}{\partial x^\sigma}\right]\delta x^\sigma + \left[g_{\mu\nu}\frac{dx^\mu}{ds} + g_{\mu\nu}\frac{dx^\nu}{ds} + 2f_\sigma\right]\frac{d}{ds}(\delta x^\sigma)}{1 - f_\mu \frac{dx^\mu}{ds}} \right\} ds = 0,$$

or, on integrating by parts, using the formula

$$\int \frac{udv}{w} = \frac{uv}{w} - \int \frac{vdu}{w} + \int \frac{uv}{w^2}\,dw,$$

and discarding the integrated part, since δx^σ vanishes at the two limits, we have

$$\int \frac{\left[g_{\mu\sigma}\frac{dx^\mu}{ds} + g_{\sigma\nu}\frac{dx^\nu}{ds} + 2f_\sigma\right] d(\delta x^\sigma)}{1 - f_\mu \frac{dx^\mu}{ds}} = -\int \frac{\frac{d}{ds}\left[g_{\mu\sigma}\frac{dx^\mu}{ds} + g_{\sigma\nu}\frac{dx^\nu}{ds} + 2f_\sigma\right]}{1 - f_\mu \frac{dx_\mu}{ds}}\,\delta x^\sigma\,ds$$
$$- \int \frac{\left[g_{\mu\sigma}\frac{dx^\mu}{ds} + g_{\sigma\nu}\frac{dx^\nu}{ds} + 2f_\sigma\right]\frac{d}{ds}\left(f_\mu \frac{dx^\mu}{ds}\right)}{\left(1 - f_\mu \frac{dx^\mu}{ds}\right)^2}\,\delta x^\sigma\,ds.$$

Downloaded by [University of Toronto Libraries] at 07:10 24 December 2014

$$\therefore \quad \frac{1}{2}\int \frac{\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}+2\frac{dx^\mu}{ds}\frac{\partial f_\mu}{\partial x^\sigma}-\frac{d}{ds}\left[g_{\mu\nu}\frac{dx^\mu}{ds}+g_{\sigma\nu}\frac{dx^\nu}{ds}+2f_\sigma\right]}{\left(1-f_\mu\frac{dx^\mu}{ds}\right)^2}\delta x^\sigma ds$$

$$-\frac{1}{2}\int \frac{\left[g_{\mu\nu}\frac{\partial x_\mu}{ds}+g_{\sigma\nu}\frac{dx_\nu}{ds}+2f_\sigma\right]\frac{d}{ds}\left(f_\mu\frac{dx^\mu}{ds}\right)}{\left(1-f_\mu\frac{\partial x^\mu}{ds}\right)^2}\delta x^\sigma ds=0\ ;$$

and since δx^σ is arbitrary, the integrand must vanish, giving

$$\frac{1}{2}\frac{\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{\partial g_\mu}{\partial x^\sigma}+2\frac{dx^\mu}{ds}\frac{\partial f_\mu}{\partial x^\sigma}-\frac{d}{ds}\left[g_{\mu\nu}\frac{dx^\mu}{ds}+g_{\sigma\nu}\frac{dx^\nu}{ds}+2f_\sigma\right]}{1-f_\mu\frac{dx^\mu}{ds}}$$

$$-\frac{1}{2}\frac{\left[g_{\mu\sigma}\frac{dx^\mu}{ds}+g_{\sigma\nu}\frac{dx^\nu}{ds}+2f_\sigma\right]\frac{d}{ds}\left(f_\mu\frac{dx^\mu}{ds}\right)}{\left(1-f_\mu\frac{dx^\mu}{ds}\right)^2}=0. \quad . \quad . \quad (4)$$

The numerator of the first fraction on the left-hand side of equation (4) equals

$$\frac{1}{2}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}+\frac{dx^\mu}{ds}\frac{\partial f_\mu}{\partial x^\sigma}-\frac{1}{2}g_{\mu\sigma}\frac{d^2x^\mu}{ds^2}-\frac{1}{2}g_{\sigma\nu}\frac{d^2x^\nu}{ds^2}$$

$$-\frac{1}{2}\frac{dx^\mu}{ds}\frac{\partial g_{\mu\sigma}}{\partial x^\nu}\frac{dx^\nu}{ds}-\frac{1}{2}\frac{dx^\nu}{ds}\frac{\partial g_{\sigma\nu}}{\partial x^\mu}\frac{dx^\mu}{ds}-\frac{\partial f_\sigma}{\partial x^\epsilon}\frac{dx^\epsilon}{ds}$$

$$=\frac{1}{2}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\left(\frac{\partial g_{\mu\nu}}{\partial x^\sigma}-\frac{\partial g_{\mu\tau}}{\partial x^\nu}-\frac{\partial g_{\sigma\nu}}{\partial x^\mu}\right)-\frac{1}{2}g_{\epsilon\tau}\frac{d^2x^\epsilon}{ds^2}$$

$$-\frac{1}{2}g_{\sigma\epsilon}\frac{d^2x^\epsilon}{ds^2}-\frac{dx^\epsilon}{ds}\left(\frac{\partial f_\sigma}{\partial x^\epsilon}-\frac{\partial f_\epsilon}{\partial x^\sigma}\right)$$

(since ϵ is a dummy suffix)

$$=-g_{\sigma\epsilon}\frac{d^2x^\epsilon}{ds^2}-[\mu\nu,\sigma]\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}-\frac{dx^\epsilon}{ds}\left(\frac{\partial f_\sigma}{\partial x^\epsilon}-\frac{\partial f_\epsilon}{\partial x^\sigma}\right),$$

which on multiplying by $g^{\sigma\alpha}$ and changing all signs becomes

$$\frac{d^2x^\alpha}{ds^2}+\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\{\mu\nu,\alpha\}+g^{\sigma\alpha}\left(\frac{\partial f_\sigma}{\partial x^\epsilon}-\frac{\partial f_\epsilon}{\partial x^\sigma}\right)\frac{dx^\epsilon}{ds}, \quad . \quad . \quad . \quad (A)$$

where $[\mu\nu,\sigma]$ and $\{\mu\nu,\alpha\}$ are Christoffel three-index symbols of the first and second kinds respectively. Similarly the numerator of the second fraction on the left-hand side of equation (4) may be written

$$-\frac{1}{2}g_{\epsilon\sigma}\left(\frac{dx^\epsilon}{ds}+g_{\sigma\epsilon}\frac{dx^\epsilon}{ds}+2f_\sigma\right)\frac{d}{ds}\left(f_\mu\frac{dx^\mu}{ds}\right)=-\left(g_{\sigma\epsilon}\frac{dx^\epsilon}{ds}+f_\sigma\right)\frac{d}{ds}\left(f_\mu\frac{dx_\mu}{ds}\right),$$

Downloaded by [University of Toronto Libraries] at 07:10 24 December 2014

which when multiplied by $-g^{\sigma\alpha}$ becomes

$$\left(\frac{dx^\alpha}{ds}+f^\alpha\right)\frac{d}{ds}\left(f_\mu\frac{dx^\mu}{ds}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(B)}$$

Combining (A) and (B) with their respective denominators, we finally have as the equations of a geodesic :

$$\frac{\dfrac{d^2x^\alpha}{ds^2}+\dfrac{dx^\mu}{ds}\dfrac{dx^\nu}{ds}\{\mu\nu,\alpha\}+g^{\sigma\alpha}\left(\dfrac{\partial f_\sigma}{\partial x^\varepsilon}-\dfrac{\partial f_\varepsilon}{\partial x^\sigma}\right)\dfrac{dx^\varepsilon}{ds}}{1-f_\mu\dfrac{dx^\mu}{ds}}+\frac{\left(\dfrac{dx^\alpha}{ds}+f^\alpha\right)\dfrac{d}{ds}\left(f_\mu\dfrac{dx^\mu}{ds}\right)}{\left(1-f_\mu\dfrac{dx^\mu}{ds}\right)^2}=0. \qquad (5)$$

§ 3. *Equations of Motion of a Charged Particle.*

We now consider the approximate equations of a geodesic by neglecting the second fraction on the left-hand side of equation (5). We have

$$\frac{d^2x^\alpha}{ds^2}+\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\{\mu\nu,\alpha\}+g^{\sigma\alpha}\left(\frac{\partial f_\sigma}{\partial x^\varepsilon}-\frac{\partial f_\varepsilon}{\partial x_\sigma}\right)\frac{dx^\varepsilon}{ds}=0. \quad . \quad . \quad . \qquad (6)$$

Or, if we set

$$f^\lambda=\frac{e}{m_0c^2}\kappa^\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

where κ^λ is the four-vector combining the vector potential[3] (F, G, H) and the scalar potential ϕ, so that

$$\kappa^\lambda=(\mathrm{F},\mathrm{G},\mathrm{H},\phi) \text{ in Galilean coordinates}, \quad . \quad . \quad . \quad . \qquad (8)$$

we have

$$\frac{\partial\kappa_\sigma}{\partial x^\varepsilon}-\frac{\partial\kappa_\varepsilon}{\partial x^\sigma}=\mathrm{F}_{\sigma\varepsilon}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

where $\mathrm{F}_{\sigma\varepsilon}$ is the electromagnetic force tensor. Equation (6) may now be written

$$\frac{d^2x^\alpha}{ds^2}+\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\{\mu\nu,\alpha\}+\frac{e}{m_0c^2}g^{\sigma\alpha}\mathrm{F}_{\sigma\varepsilon}\frac{dx^\varepsilon}{ds}=0,$$

or

$$\frac{d^2x^\alpha}{ds^2}+\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\{\mu\nu,\alpha\}+\frac{e}{m_0c^2}\mathrm{F}_\varepsilon{}^\alpha\frac{dx^\varepsilon}{ds}=0, \quad . \quad . \quad . \quad . \qquad (10)$$

which represents the equations of motion[4] of a charged particle of charge e and mass m_0 when the radiation terms are neglected. We reserve a discussion of the radiation terms, represented by the second fraction on the left-hand side of (5), for another occasion.

§ 4. *Nature of the New Metric, Magnitude of a Vector and Angle between Two Vectors.*

Let l^μ be a contravariant vector, the magnitude l of the vector in Riemannian geometry is determined by

$$g_{\mu\nu}l^\mu l^\nu=l^2,$$

Downloaded by [University of Toronto Libraries] at 07:10 24 December 2014

Downloaded by [University of Toronto Libraries] at 07:10 24 December 2014

so that l has two equal and opposite values. Assuming these to be real, the negative value is discarded and thus

$$l=\sqrt{g_{\mu\nu}l^{\mu}l^{\nu}}.$$

We follow a similar procedure with the new metric. The magnitude l, assumed to be real, of a contravariant vector l^{μ} is now the positive root of the equation

$$g_{\mu\nu}l^{\mu}l^{\nu}+2f_{\mu}l^{\mu}l-l^2=0, \quad \ldots \quad (11)$$

so that

$$l=\sqrt{g_{\mu\nu}l^{\mu}l^{\nu}+(f_{\mu}l^{\mu})^2}+f_{\mu}l^{\mu}. \quad \ldots \quad (12)$$

Equation (12) may be written

$$l=\sqrt{g^{\mu\nu}l_{\mu}l_{\nu}+(f^{\mu}l_{\mu})^2}+f^{\mu}l_{\mu}, \quad \ldots \quad (13)$$

so that l is also the magnitude of the associated covariant vector l_{μ}. We observe that if $l=0$, then we have from (11)

$$g_{\mu\nu}l_{\mu}l_{\nu}=0 \text{ (for a null vector)}, \quad \ldots \quad (14)$$

which is the same as in Riemennian geometry. Similarly, if we put $ds=0$ in equation (2), we have

$$g_{\mu\nu}dx^{\mu}dx^{\nu}=0 \text{ (for } ds=0), \quad \ldots \quad (15)$$

so that the path of a ray of light remains the same as in Einstein's theory. Let l^{μ} and l'^{ν} be two contravariant vectors ; to define the angle θ between them two different lines of procedure suggest themselves. Either we write

$$\cos\theta=\frac{g_{\mu\nu}l^{\mu}l'^{\nu}}{ll'}+2f_{\mu}\frac{l^{\mu}+l'^{\mu}}{l+l'}. \quad \ldots \quad (16.1)$$

or we solve the equation

$$\frac{g_{\mu\nu}l^{\mu}l'^{\nu}}{ll'}+2f_{\mu}\frac{l^{\mu}+l'^{\mu}}{l+l'}\sqrt{\cos\theta}-\cos\theta=0 \quad \ldots \quad (16.2)$$

for $\sqrt{\cos\theta}$ and discard the negative value, so that

$$\sqrt{\cos\theta}=\sqrt{\frac{g_{\mu\nu}l^{\mu}l'^{\nu}}{ll'}+\left[\frac{f_{\mu}(l^{\mu}+l'^{\nu})}{l+l'}\right]^2}+\frac{f_{\mu}(l^{\mu}+l'^{\mu})}{l+l'}. \quad \ldots \quad (17.2)$$

The latter alternative leaves the condition for perpendicularity of two vectors the same as in Riemannian geometry. Thus

$$g_{\mu\nu}l^{\mu}l'^{\nu}=0 \quad \text{(for two orthogonal non-null vectors)}. \quad \ldots \quad (18.2)$$

§ 5. *Mass of a Particle, Dependence of Mass on the Potential Energy of the Particle.*

If in equation (2) we adopt Galilean values for the tensor $g_{\mu\nu}$ and use Galilean coordinates (x, y, z, t), we have, using (7) and (8),

$$\frac{1}{c}\frac{ds}{dt}=\pm\sqrt{1-\frac{u^2}{c^2}+\frac{e^2}{m_0{}^2c^4}\left(\phi-\frac{\mathrm{F}\dot{x}+\mathrm{G}\dot{y}+\mathrm{H}\dot{z}}{c}\right)^2+\frac{e}{m_0c^2}\left(\phi-\frac{\mathrm{F}\dot{x}+\mathrm{G}\dot{y}+\mathrm{H}\dot{z}}{c}\right)}, \quad . \quad . \quad (19)$$

where u is the velocity. So that if the mass m is defined by the relation

$$\frac{m}{m_0}=\frac{cdt}{ds}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and we assume m to be positive, we have

$$m=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}+\frac{e^2}{m_0{}^2c^4}\left(\phi-\frac{\mathrm{F}\dot{x}+\mathrm{G}\dot{y}+\mathrm{H}\dot{z}}{c}\right)^2+\frac{e}{m_0c^2}\left(\phi-\frac{\mathrm{F}\dot{x}+\mathrm{G}\dot{y}+\mathrm{H}\dot{z}}{c}\right)}}. \quad . \quad . \quad . \quad . \quad (21)$$

If $\frac{u}{c}$ and $\frac{e\phi}{m_0c^2}$

are small, we have, to the first order,

$$m=m_0\left(1-\frac{e\phi}{m_0c^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

This offers an explanation of the deficiencies in the masses of atomic nuclei known as the "packing" effect. It also provides a quantitative measure of nuclear fields.

§ 6. *Application to Nuclear Fields.*

Consider a nucleus containing Z protons and (A—Z) neutrons, so that Z is the atomic number and A the mass number. If M is the atomic weight of the nucleus on the physical ^{16}O scale, we have

$$\mathrm{M}=\mathrm{M}_n(\mathrm{A}-\mathrm{Z})+\mathrm{M}_p\mathrm{Z}(1-k),$$

where M_n and M_p are the atomic weights of a neutron and a proton respectively, and

$$k=\frac{e\phi}{m_0c^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

m_0 being now the free rest mass of a proton. Taking [5] for M_n the value 1·00893, and [6] for M_p the value 1·00813, we have

$$\mathrm{M}-\mathrm{A}=\cdot 00893\mathrm{A}-\cdot 00080\mathrm{Z}-1\cdot 00813\mathrm{Z}k. \quad . \quad . \quad . \quad . \quad (24)$$

The following table gives the values of k calculated from equation (24), using determinations of M given by Livingston and Bethe [7] and by G. Gamow [8] based on mass spectrograph measurements by Aston [9], by Bainbridge and Jordan [10] and by Mattauch [11] and correlated with the data accruing from nuclear reactions to secure "best" values :—

Downloaded by [University of Toronto Libraries] at 07:10 24 December 2014

Nucleus	A	Z	M–A	Authors	k	ϕ in millions of volts	r_0 in 10^{-13} cm.
^{2}H	2	1	0·014 73	L.B.	0·002 31	2·168	0·664
^{3}H	3	1	0·017 10	G.	0·008 82	8·273	0·174
^{3}Hl	3	2	0·017 10	G.	0·004 01	3·764	0·765
^{4}Hl	4	2	0·003 89	L.B.	0·014 99	14·07	0·405
^{6}Li	6	3	0·016 70	G.	0·011 40	10·70	0·404
^{7}Li	7	3	0·018 18	L.B.	0·013 86	13·01	0·332
^{8}Be	8	4	0·007 80	G.	0·014 99	14·06	0·410
^{9}Be	9	4	0·015 16	L.B.	0·015 38	14·43	0·399
^{10}Be	10	4	0·016 31	L.B.	0·017 31	16·24	0·355
^{10}B	10	5	0·016 10	G.	0·013 73	12·88	0·559
^{11}B	11	5	0·012 80	G.	0·016 15	15·16	0·475
^{12}C	12	6	0·003 98	L.B.	0·016 26	15·26	0·566
^{13}C	13	6	0·007 61	L.B.	0·016 74	15·71	0·550
^{14}N	14	7	0·007 50	L.B.	0·015 86	14·88	0·677
^{15}N	15	7	0·004 89	L.B.	0·017 49	16·41	0·614
^{16}O	16	8	0·000 00	—	0·016 92	15·88	0·725
^{17}O	17	8	0·004 60	G.	0·017 46	16·38	0·703
^{18}O	18	8	0·003 69	L.B.	0·018 68	17·52	0·657
^{19}F	19	9	0·004 52	L.B.	0·017 41	16·33	0·793
^{20}Ne	20	10	$\bar{1}$·998 81	L.B.	0·017 04	15·99	0·901
^{21}Ne	21	10	$\bar{1}$·999 68	L.B.	0·017 84	16·74	0·860
^{22}Ne	22	10	$\bar{1}$·998 64	L.B.	0·018 83	17·16	0·815
^{27}Al	27	13	$\bar{1}$·990 90	G.	0·018 30	17·17	1·09
^{23}Si	28	14	$\bar{1}$·986 80	L.B.	0·017 74	16·65	1·12
^{29}Si	29	14	$\bar{1}$·986 60	L.B.	0·018 51	17·37	1·16
^{31}P	31	15	$\bar{1}$·984 10	L.B.	0·018 57	17·42	1·24
^{32}S	32	16	$\bar{1}$·982 30	L.B.	0·018 02	16·91	1·36
^{35}Cl	35	17	$\bar{1}$·981 30	L.B.	0·018 53	17·39	1·41
^{37}Cl	37	17	$\bar{1}$·978 80	L.B.	0·019 72	18·50	1·32
^{36}A	36	18	$\bar{1}$·978 00	L.B.	0·018 15	17·03	1·52
^{40}A	40	18	$\bar{1}$·975 04	L.B.	0·020 26	19·01	1·36
^{45}Si	45	21	$\bar{1}$·968 00	G.	0·019 70	18·48	1·64
^{52}Cr	52	24	$\bar{1}$·948 00	G.	0·020 13	18·89	1·83
^{58}Ni	58	28	$\bar{1}$·942 00	G.	0·019 61	18·40	2·19
^{64}Zu	64	30	$\bar{1}$·937 00	G.	0·020 89	18·94	2·28
^{75}As	75	33	$\bar{1}$·934 00	G.	0·021 32	20·00	2·38
^{78}Se	78	34	$\bar{1}$·938 00	G.	0·021 33	20·02	2·45
^{80}Se	80	34	$\bar{1}$·941 00	G.	0·021 77	20·42	2·40
^{79}Br	79	35	$\bar{1}$·929 00	G.	0·021 22	19·90	2·53
^{81}Br	81	35	$\bar{1}$·926 00	G.	0·021 81	20·46	2·46
^{78}Kr	78	36	$\bar{1}$·926 00	G.	0·020 44	19·18	2·70
^{80}Kr	80	36	$\bar{1}$·926 00	G.	0·020 93	19·64	2·64
^{82}Kr	82	36	$\bar{1}$·927 00	G.	0·021 40	20·07	2·58
^{83}Kr	83	36	$\bar{1}$·927 00	G.	0·021 64	20·30	2·55
^{84}Kr	84	36	$\bar{1}$·928 00	G.	0·021 86	20·51	2·53
^{86}Kr	86	36	$\bar{1}$·929 00	G.	0·022 33	20·95	2·47
^{93}Nb	93	41	$\bar{1}$·926 00	G.	0·021 09	19·79	2·98
^{100}Mo	100	42	$\bar{1}$·945 00	G.	0·021 60	20·26	2·98
^{103}Rh	103	45	$\bar{1}$·920 00	G.	0·021 24	19·93	3·25

Downloaded by [University of Toronto Libraries] at 07:10 24 December 2014

Nucleus	A	Z	M–A	Authors	k	Q in millions of volts	r_0 in 10^{-13} cm.
^{120}Sn	120	50	$\bar{1}$·912 00	G.	0·022 21	20·84	3·45
^{127}I	127	53	$\bar{1}$·932 00	G.	0·021 64	20·30	3·76
^{134}Xe	134	54	$\bar{1}$·929 00	G.	0·022 49	21·10	3·68
^{133}Cs	133	55	$\bar{1}$·933 00	G.	0·021 83	20·48	3·87
^{138}Bu	138	56	$\bar{1}$·916 00	G.	0·022 52	21·13	3·82
^{181}Ta	181	73	$\bar{1}$·928 00	G.	0·022 15	20·78	5·06
^{187}Re	187	75	$\bar{1}$·981 00	G.	0·021 54	20·21	5·34
^{190}Os	190	76	$\bar{1}$·980 00	G.	0·021 61	20·28	5·40
^{192}Os	192	76	$\bar{1}$·980 00	G.	0·021 85	20·50	5·34
^{200}Hg	200	80	0·016 00	G.	0·021 15	19·85	5·80
^{203}Tl	203	81	0·037 00	G.	0·020 95	19·66	5·93
^{205}Tl	205	81	0·037 00	G.	0·021 17	19·86	5·87

The data extend from Z=1 to Z=81 for 61 different nuclei of 37 different elements. The value of ϕ in millions of volts is calculated from the formula

$$\phi=\frac{m_0c^3k}{e}\times10^{-14}\text{ million volts,} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and the effective radius r_0 from the equation

$$r_0=\frac{Ze^2}{m_0c^2k}\text{ cm.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

with the following data quoted by Taylor and Glasstone [12]:—

$$m_0=1{\cdot}6725\times10^{-24}\text{ gm., } c=2{\cdot}9978\times10^{10}\text{ cm. sec.}^{-1},$$

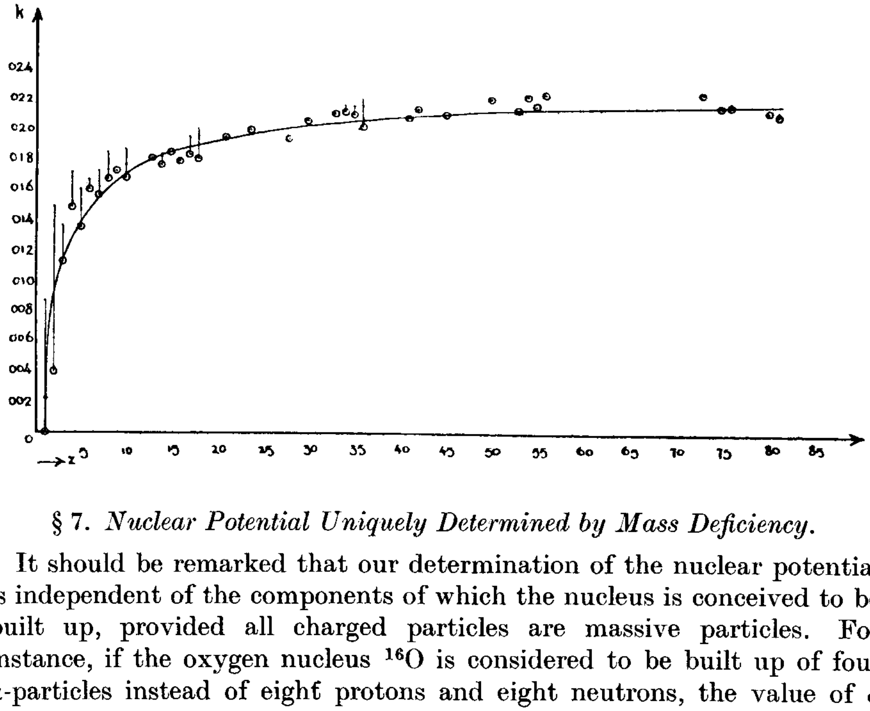
$$e=4{\cdot}8025\times10^{-10}\text{ abs. e.s.u.}$$

The abbreviations L.B. stand for Livingston and Bethe, and G. for Gamow. With regard to the last element in the table, thallium, two isotopes are included in the data, namely ^{203}Tl and ^{205}Tl, and their effective radii are $5{\cdot}93\times10^{-13}$ cm. and $5{\cdot}87\times10^{-13}$ cm. respectively. The atomic number of thallium being 81, it is possible to compare our results with the results obtained by G. Gamow [15] in his theory of α-disintegration, which gives a satisfactory explanation of the Geiger-Nuttall relation between the range and the decay constant of α-radiators. Gamow obtains three values for nuclei of atomic number 81, namely $r_0=5{\cdot}7\times10^{-13}$ cm., $r_0=6{\cdot}0\times10^{-13}$ cm. and $r_0=6{\cdot}3\times10^{-13}$ cm., which are seen to be in rather striking agreement with our results.

In the diagram k is plotted against Z. The points with small circles round them correspond to the lowest or "ground" potential levels for each element as given by the available data. Other values, corresponding to different isotopes of the same element, are represented by small dots. The different levels for the same element are joined by a line parallel to the axis of k. The curve is drawn to represent the change in "ground"

Downloaded by [University of Toronto Libraries] at 07:10 24 December 2014

values of k with Z. It may prove possible with fuller data to draw a series of curves through the higher potential values, but owing to the insufficiency of the present data, this could not be done with any degree of certainty. In fact, some of the values which, being the lowest for each element, are now assumed to be " ground " levels may prove to be otherwise. The data are, however, just enough to indicate, unmistakably, the general trend of the curve, which is seen to rise steeply at first, then more gradually, reaching a value corresponding to about 20 million volts at Z=81.



§ 7. *Nuclear Potential Uniquely Determined by Mass Deficiency.*

It should be remarked that our determination of the nuclear potential is independent of the components of which the nucleus is conceived to be built up, provided all charged particles are massive particles. For instance, if the oxygen nucleus ^{16}O is considered to be built up of four α-particles instead of eight protons and eight neutrons, the value of ϕ and the mass of the nucleus are unaffected, since we must take into consideration the diminution in mass of each α-particle arising from its presence in a field of a higher potential than its own nuclear field. If k_0 is the value of k for ^{16}O and k_α for ^{4}He, the following equation which expresses the building up of ^{16}O from four ^{4}He nuclei, namely

$$16{\cdot}00000 = 4 \times 4{\cdot}00389 \left[1 - \frac{2 \times 1{\cdot}00813}{4{\cdot}00389} (k_0 - k)_\alpha \right], \quad . \quad . \quad (27)$$

is seen to be satisfied if we substitute the values given in the table, namely $k_0 = {\cdot}01692$ and $k_\alpha = {\cdot}01499$. The same is true for neutrons or other massive components. Thus the nuclear potential is uniquely determined by its mass deficiency, in contrast with certain current theories which make the so-called " binding energy " of the nucleus dependent on the proportion of protons, α-particles, etc. of which it is conceived to be built up.

Downloaded by [University of Toronto Libraries] at 07:10 24 December 2014

Downloaded by [University of Toronto Libraries] at 07:10 24 December 2014

§ 8. *Non-massive Particles.*

For non-massive particles the position is different. If μ_0 is the free rest mass of an electron, then $\frac{e\phi}{c^2}$ is large compared with μ_0 for nuclear fields. For a field of 20 million volts $\frac{e\phi}{\mu_0 c^2}$ is equal to about 40. If $\frac{\mu_0 c^2}{e\phi}$ is treated as a small quantity, equation (21) now gives to the first order

$$\mu = \frac{2e\phi}{c^2}, \qquad (28)$$

so that the mass μ of the electron becomes proportional to the potential of the field. For a positron the mass is diminished, being divided by $\frac{2e\phi}{\mu_0 c^2}$. Thus the mass of a positron in a field of 20 million volts is equal to about one-eightieth part of its free rest mass. A positron would accordingly dwindle, so to speak, to a very small mass in the neighbourhood of nuclei. This may be related to the observed short-livedness of the positron. Equation (28) is in one respect quite remarkable. The mass of an electron in a strong field seems to be quite independent of its free mass. Of course, this is only a first approximation, but it is nevertheless noteworthy. The mass of the electron appears to be completely " dissolved " in the field. The ejection of an electron from a nucleus becomes therefore more of the nature of an emission of field energy than an ejection of a particle. This is in agreement with the observed facts of β-disintegration.

Summary.

A new metric of space-time is introduced in the theory of relativity which is a generalized form of the Riemannian metric. The world-line of a charged particle is shown to be a geodesic, and the equations of motion of the particle are obtained. The metric gives the mass of a particle as a function of its potential energy. The mass deficiency of nuclei is thus explained as arising from the presence of their charged components in the nuclear field. The potential of nuclear fields is determined from exact mass measurements for 37 different elements. Calculations of effective radii agree with results obtained by Gamow from radioactive data. The mass of an electron in a nuclear field indicates that β-disintegration is more akin to emission of field energy than to particle ejection. The mass of a positron becomes very small in the neighbourhood of nuclei. A curve is drawn from the data to represent the growth of potential with atomic number Z, reaching a value of about 20 million volts at Z=81.

References.

(1) H. Weyl, ' Space—Time—Matter,' London (1922).
(2) M. Kaluza, *Sitzungsber. d. Berl. Akad.* p. 966 (1921).
(3) See Eddington, ' The Mathematical Theory of Relativity,' p. 171.

(4) See Eddington, *loc. cit.* p. 190; also G. Schott, 'Electromagnetic Radiation,' p. 283, § 278, Cambridge (1912).
(5) See H. A. Bethe, Phys. Rev. liii. p. 313 (1938).
(6) See J. Mattauch, Phys. Rev. lvii. p. 1115 (1940).
(7) M. S. Livingston and H. A. Bethe, Rev. Mod. Phys. ix. p. 336 (1937), quoted by Taylor and Glastone, reference below, p. 53.
(8) G. Gamow, 'Structure of Atomic Nuclei and Nuclear Transformations,' p. 249 (Oxford, 1937).
(9) F. W. Aston, 'Nature,' cxxxv. p. 541 (1935); cxxxvii. pp. 357, 613 (1936); cxxxviii. p. 1094 (1936); cxxxix. p. 922 (1937). 'Mass Spectra and Isotopes,' (London, 1933).
(10) K. T. Bainbridge and E. B. Jordan, Phys. Rev. xlix. p. 883 (1936); l. p. 98 (1936); li. pp. 384, 385 (1937).
(11) *Loc. cit.*
(12) H. S. Taylor and S. Glasstone, 'Atomistics and Thermodynamics,' New York, p. 671 (1943), quoted from R. T Birge
(13) *Loc. cit.* p. 105, Table V.

LXXXVII. *Waves in Deep Water due to a concentrated Surface Pressure.*

By George Green, Lecturer in Physics in Glasgow University*.

[Received September 1, 1947.]

The problems of determining the wave-systems arising from the application of a stationary periodic, or moving surface pressure, might be regarded as fundamental problems of Hydrodynamics, in view of their relation to the important problem of Ship-Waves. Solutions of the problem of motion due to a stationary periodic surface pressure are given by Lamb in Edition 3 of 'Hydrodynamics,' pp. 375–7, and in his Presidential Address "On Deep Water Waves" to the London Mathematical Society, Nov. 10 (1904). Edition 6 of 'Hydrodynamics' does not include this problem, but it contains a full account of the related problem of the motion due to a moving surface pressure. The method employed to obtain a determinate result is the same in both cases, but these solutions are not in agreement one with the other.

In Edition 6, p. 406, reference is given to "a different treatment of the latter problem by Lord Kelvin in 'Deep Water Ship Waves,'" Proc. R.S.E. XXV. p. 562 (1905). In a still later paper, Proc. R.S.E. XXVI. p. 412 (1906), Lord Kelvin applied yet another different treatment to a group of cognate problems involving surface pressure, but the group did not include the case of a concentrated pressure which is that dealt with by Lamb. Some years ago the present writer had occasion to use Professor Lamb's result for the case of the stationary periodic pressure in connection with a practical problem, and found difficulty with it. He has thus been led to work out the solution of both problems referred to above by the method used in Lord Kelvin's last "Waves" paper,

* Communicated by the Author.

Downloaded by [University of Toronto Libraries] at 07:10 24 December 2014